# BOUNDARY LAYER AND VELOCITY DISTRIBUTION

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This chapter is devoted to the vertical velocity distribution of the open-channel flow. First, the concept of the boundary layer is introduced. The development of the boundary layer is described and various definitions of the boundary layers are introduced. Then the division of open-channel flows are given. For open-channel flows, two distinct boundaries exit, namely bed and free surface. Depending on their relative dominance, flow regions are divided. Finally, velocity distribution is derived considering the characteristics of the wall. Mainly. the mixing length theory is used and discussed.

## 1. Boundary Layer

### 1.1 Definition

The figure below shows the development of the boundary layer (BL) when water enters a channel ideally. In the channel, the effect of the roughness on the velocity distribution is indicated by the upper curve. Outside this curve, the velocity distribution is uniform. The region inside this curve is the BL with thickness $\delta$. A common definition of the BL thickness is the normal distance from the surface at which the velocity *u* is equal to 99% of the limiting velocity *U*.

The flow within the BL begins as a laminar flow without regard to the state of the approach flow. As the BL grows along the surface, a transition occurs and the flow within the BL becomes turbulent. Regardless of whether the entering flow is laminar or turbulent, the transition occurs. But, if the entering flow is highly turbulent, the transition occurs in the region very close to the leading edge.

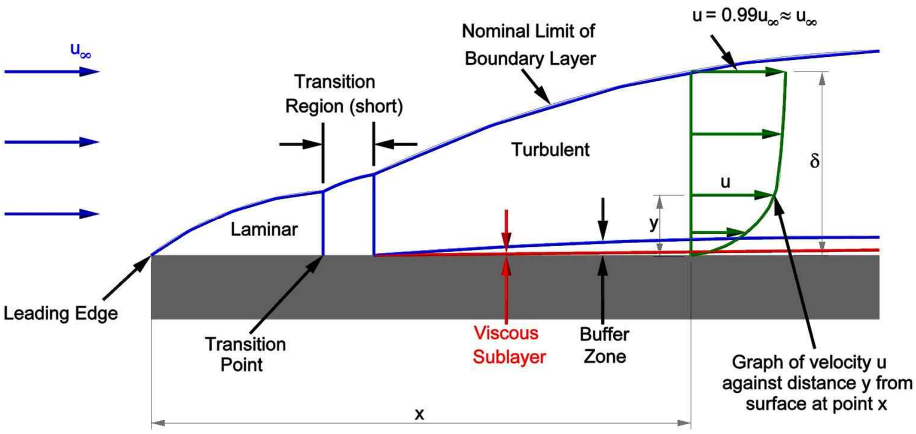



Figure 1. Development of boundary layer in the open-channel flow

## 1.2 Boundary Layer Thicknesses

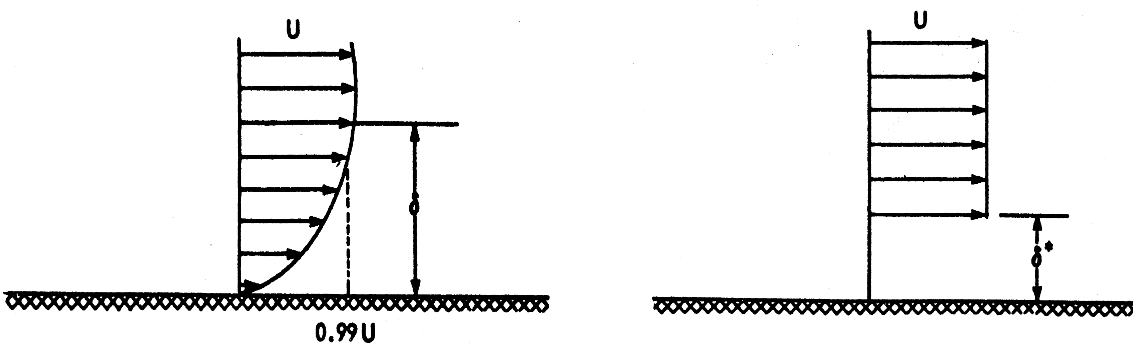
The effect of the BL on the flow is a fictitious upward displacement of a channel bottom to a virtual position by an amount equal to the BL thickness. For the mass, the displacement is defined by

$$\delta_1 = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dz \tag{1}$$

which represents the amount that the thickness of the body must be increased so that the fictitious uniform inviscid flow has the same mass flowrate as the actual viscous flow. Similarly, for the momentum and energy, the momentum thickness and energy thickness are given, respectively, by

$$\delta_2 = \int_0^{\delta} \left(1 - \frac{u}{U}\right) \frac{u}{U} dz \tag{2}$$

$$\delta_3 = \int_0^{\delta} \left(1 - \frac{u^2}{U^2}\right) \frac{u}{U} dz \tag{3}$$




(a) boundary-layer thickness (b) displacement thickness

Figure 2. Boundary layer thickness and displacement thickness

## 2. Close-to-Wall Hydraulics

### 2.1 Division of the Open-Channel Flow

The figure below shows the division of the open-channel flow. The flow depth can be divided into three regions, namely the free surface region, intermediate region, and wall region. In the free surface region, the characteristic length and velocity are the flow depth and maximum velocity, respectively, and, in the wall region, they are $\nu / u_*$ and $u_*$, respectively.

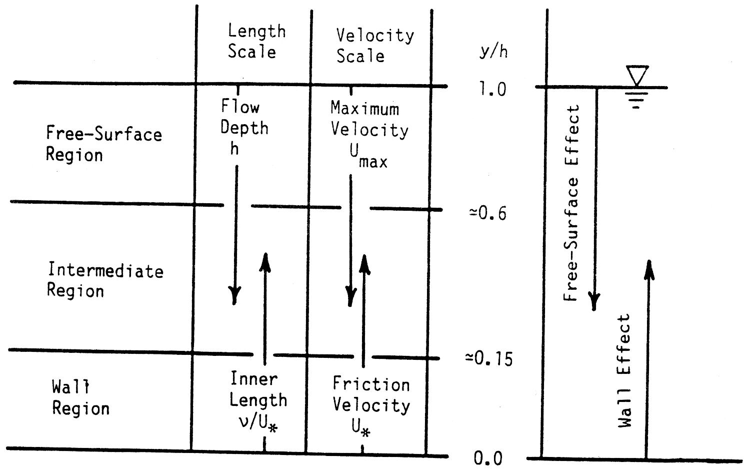



Figure 3. Subdivision of open-channel flow (Nezu and Nakagawa, 1993)

### 2.2 Subdivision of Wall Region

Depending upon the distribution of viscous and turbulent shear stresses, the boundary layer region is divided into the following three zones:

(1) viscous region ( $0 \le zu_* / \nu \le 30$ )

1) the viscous sublayer ($1 \le zu_* / v \le 5$) where turbulent shear stress is negligible compared with molecular momentum transfer

2) the buffer layer where both viscous and turbulent stresses are important

(2) overlap region ($30 \le zu_* / v$ and $z \le 0.2\delta$)

In this region, turbulence production and dissipation are locally balanced

(3) wake region

This region is characterized by diminishing Reynolds stresses.

From turbulence measurements, two distinct regions can be distinguished: the inner region, near the wall, where the logarithmic velocity distribution is valid, and the outer region where the velocity profile deviates slightly, but systematically, from the logarithmic law.

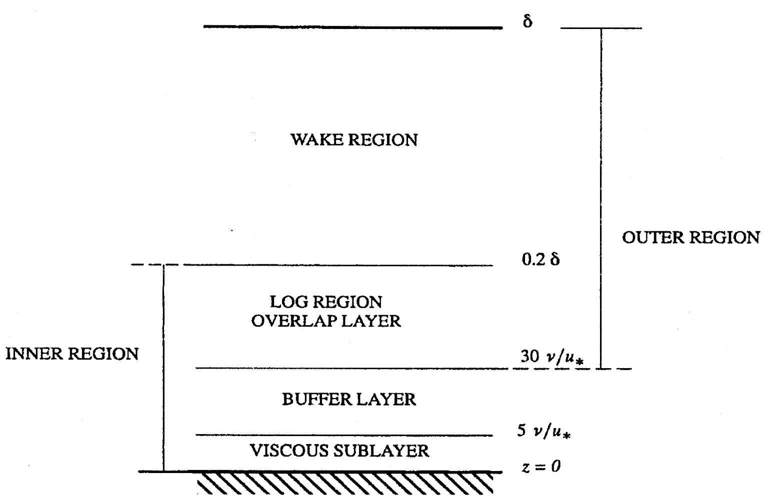



Figure 4. Different flow regions in smooth wall-bounded shear flow

### 2.3 Velocity Distribution in Turbulent Boundary Layer

(1) Inner Layer (Law of the Wall)

The inner layer consists of viscous sublayer, buffer layer, and inertial sublayer. The velocity in the inner layer is given by

$$\frac{u}{u_*} = f(z^+, k^+) \tag{4}$$

where $z^+ = zu_* / \nu$ and $k^+ = ku_* / \nu$ .

(2) Outer Layer (Velocity Defect Law)

The velocity in the outer layer is given by

$$\frac{U - u}{u_*} = F(\eta, H) \tag{5}$$

where $\eta = z / \delta$ and $H = \delta_1 / \delta_2$ .

## 3. Velocity Distribution of Turbulent Flows

According to Newton's law of viscosity, the shear stress is related to strain rate. This can be applied to laminar flows. By analogy, the shear stress for the turbulent flow is expressed by the sum of shear stresses due to fluid intrinsic viscosity and due to turbulence. That is,

$$\tau = (\mu + \mu_T)\frac{d\bar{u}}{dz} \tag{6}$$

where $\mu$ is the dynamic viscosity and $\mu_T$ is the turbulent viscosity (or dynamic eddy viscosity). The turbulent shear stress is given by

$$\tau_T = -\rho\overline{u^{'}w^{'}} \tag{7}$$

In a gas, one molecule travels an average distance before striking another. This distance is known as *the mean free path*. Using this as an analogy, Prandtl assumed that a fluid particle moves a distance *l* without changing its momentum by the new environment.

In Prandtl's theory (1925), expressions for *u'* and *w'* are obtained in terms of a mixing length *l* and the velocity gradient $d\bar{u}/dz$. From dimensional analysis, the velocity fluctuation can be expressed by

$$u' = w' = l\frac{d\bar{u}}{dz} \tag{8}$$

Therefore, one obtain

$$\tau = \rho l^2\left(\frac{d\bar{u}}{dz}\right)^2 \tag{9}$$

where the kinematic eddy viscosity is

$$v_T = l^2\frac{d\bar{u}}{dz} \tag{10}$$

Note that $v_T$ is no longer a fluid property.

The particular relationship of *l* to wall distance is not given by Prandtl's theory. von Karman proposed

$$l = \kappa \frac{d\bar{u}/dz}{d^2\bar{u}/dz^2} \tag{11}$$

where $\kappa$ is von Karman constant in turbulent flows regardless of the boundary or Reynolds number. Prandtl made the following assumptions <u>for the region near the wall</u>: (1) the mixing length is proportional to the distance from the wall (the constant proportionality is in fact von Karman constant): $l = \kappa z$, and (2) shear stress is almost constant: $\tau = \tau_0 = \text{constant}$. From eq.(14), we have

$$d\bar{u} = \sqrt{\frac{\tau_0}{\rho}} \frac{1}{\kappa} \frac{dz}{z} \tag{12}$$

Integration of the above equation yields

$$\bar{u} = \frac{1}{\kappa} u_* \ln\left(\frac{z}{z_0}\right) \tag{13}$$

where $z_0$ is an integration constant. Eq.(13) states that the velocity distribution is logarithmic in the close-to-wall region.

When the surface is smooth, $z_0$ in Eq.(13) has been found to depend on the friction velocity and kinematic viscosity. That is,

$$z_0 = \frac{m\nu}{u_*} \tag{14}$$

From Nikuradse's experimental data on smooth pipes, $m$ is about 1/9. When the surface is rough, $z_O$ is a function of the roughness height, i.e.,

$$z_0 = mk \tag{15}$$

where $m$ is about 1/30. This value also came from Nikuradse's experimental data on rough pipes, and $k$ stands for the mean diameter in sand grains. Thus the respective logarithmic laws for smooth and rough surfaces are

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln\frac{u_* z}{\nu} + 5.5 \qquad \text{for smooth boundary} \tag{16}$$

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln\frac{z}{k_s} + 8.5 \qquad \text{for rough boundary} \tag{17}$$

where $k_s$ is the effective roughness height.

The figure below shows the change of the mixing length with the vertical distance from the bed. Measured data in Yang and Choi (2005) show that the mixing length increases with the distance for $z < 0.6$ and decreases slightly thereafter, regardless of the bed roughness. The ramp function in the figure denotes the approximation of the mixing length with $\kappa$ = 0.41 and $\beta$ = 0.12, as proposed by Nezu and Rodi (1986). The measured data yields $\kappa$ = 0.34 and $\beta$ = 0.14 and 0.13 for smooth and rough beds, respectively. The measured data indicates that the mixing length decreases near the free surface. This is reasonable, as discussed in Nezu and Nakagawa (1996), because the water surface restricts the size of the turbulent eddies, thus reducing the turbulent length scale near the free surface.

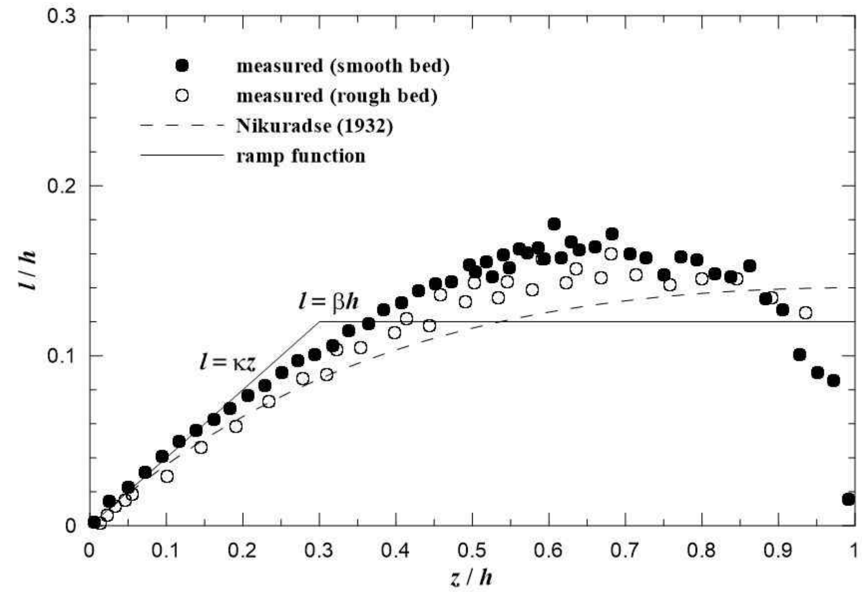



Figure 5. Mixing length versus distance from the bed

## References

Nezu, I. and Nakagawa, H. (1996). Turbulence in Open-Channel Flows, IAHR Monograph, Balkema, Rotterdam, The Netherland.

Yang, W. and Choi, S. (2005). Turbulence measurements of open-channel flows with Laser-Doppler anemometer, Journal of the Korean Society of Civil Engineers, 25(2B), 123-134 (in Korean).

### Problems

1. Using the following velocity profile, obtain the BL thickness, momentum thickness, energy thickness, and bed shear stress $\tau_0$ for flow over a flat surface:

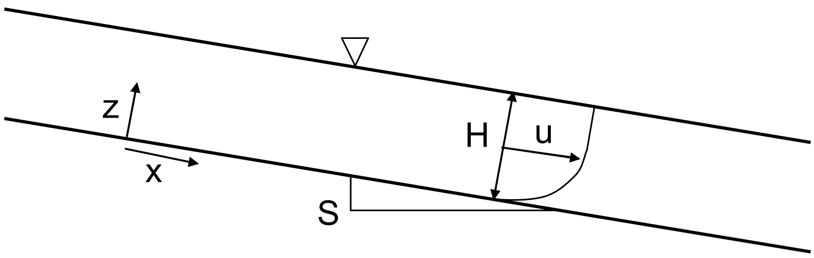
$$\frac{u}{U} = a + b\frac{z}{\delta}$$

2. Repeat Problem 1 with the following velocity profile:

$$\frac{u}{U} = \sin\left(\frac{\pi}{2}\frac{z}{\delta}\right)$$

3. The illustrated open channel has a slope angle $\alpha$ low enough to approximate $\tan\alpha \cong S$. It contains a Bingham fluid with density $\rho_m$, dynamic viscosity $\mu_m$ and yield strength $\tau_{yield}$. Here $\rho_m > \rho$ and $\mu_m > \mu$, where $\rho$ and $\mu$ are the corresponding values for water. The flow is steady and uniform in the x and y directions (where y is out of the page), and v = 0. For this flow the constitutive relation reduces to the following form:

$$\mu_m \frac{du}{dz} = \begin{cases} 0 & , \quad \tau \le \tau_{yield} \\ \tau - \tau_{yield} & , \quad \tau > \tau_{yield} \end{cases}$$




(1) Show that the distribution for shear stress is exactly the same as that for the case of a Newtonian fluid:

$$\tau = \tau_b\left(1 - \frac{z}{H}\right) \quad , \quad \tau_b = \rho_m gHS$$

where H denotes the (constant) depth of flow.

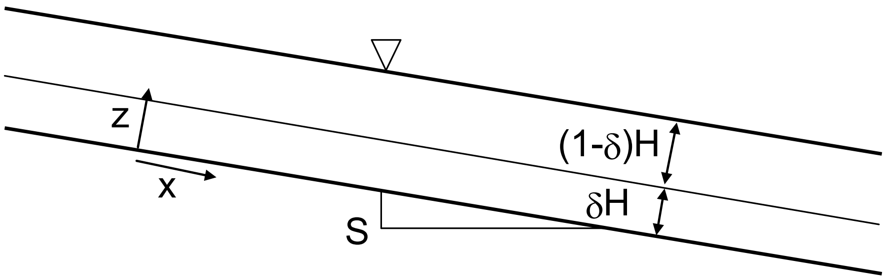
(2) Show that if $\tau_b \le \tau_{yield}$ there is no flow. Derive the form for the velocity profile in the case that $\tau_b > \tau_{yield}$. Determine forms for $U_s/U$ and $U/u_*$, where $U_s$ denotes surface velocity (at z = H) and

$$u_* = \sqrt{\frac{\tau_b}{\rho_m}} = \sqrt{gHS}$$

(3) Consider a case for which $\rho_m$ = 1700 kg/m$^3$, $\mu_m$ = 1.5 Pa s and $\tau_{yield}$ = 400 Pa. (The corresponding values for water at 20°C are $\rho$ = 1000 kg/m$^3$, $\mu$ = 0.001 Pa s = Pa s and $\tau_{yield}$ = 0). The slope of the channel S is 0.05. What is the minimum depth of mud for a flow to occur? What is the surface flow velocity $u_s$ and the depth-averaged flow velocity U for flow depths that are 1.1x, 1.25x and 1.5x this minimum depth?

4. Now consider a case for which fraction $\delta$ of the total depth of flow consists of water (bottom layer) and fraction $(1 - \delta)$ consists of mud, all with the properties listed above.

(1) Derive relations for the shear stress $\tau$, the flow velocity u and the parameters $u_s/U$ and $U/u_*$ .




(2) Let the total flow depth be 1.2x the minimum flow depth for the case of Problem 3(3), and the bed slope be the same as Problem 3(3). In addition, $\delta = 0.02$. Compare the values of $u_s$ and U for this case with the corresponding values for that of Problem 3(3), for which $\delta = 0$ (no lower water layer). Use the numerical values of Problem 3(3) to do this.